COMPARISON OF GALOIS ORBITS OF SPECIAL POINTS OF SHIMURA VARIETIES

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Let $(G,\mathcal{X})$ be a Shimura datum satisfying

$$\text{(SV5)} \qquad \begin{gathered} Z(G)^\circ \text{ is an almost direct product of a } \mathbb{Q}\text{-split torus } Z_G^s \\ \text{with a torus of compact type } Z_G^c \text{ defined over } \mathbb{Q} \end{gathered}$$

In this case, G is an almost direct product of Z_G^s with $G^c := Z_G^c G^{\mathrm{der}}$. Let $E = E(G,\mathcal{X})$ be its reflex field and let $K' = \prod_p K'_p \subset K = \prod_p K_p$ be two neat open compact subgroups of $G(\mathbb{A}_f)$. We have a natural morphism

$$\rho\colon \mathrm{Sh}_{K'}(G,\mathcal{X}) \to \mathrm{Sh}_K(G,\mathcal{X}). \tag{1}$$

By [2, Theorem 5.5, Proposition 5.2], $\mathrm{Sh}_{K'}(G,\mathcal{X})$, $\mathrm{Sh}_K(G,\mathcal{X})$ and ρ are all defined over E.

Let s be a special point of $\mathrm{Sh}_{K'}(G,\mathcal{X})$, then $s \in \mathrm{Sh}_{K'}(G,\mathcal{X})(\overline{E})$. The goal of this section is to compare $|\operatorname{Gal}(\overline{E}/E)s|$ and $|\operatorname{Gal}(\overline{E}/E)\rho(s)|$. Let $T := \mathrm{MT}(s)$ be the Mumford-Tate group of s. Define $K'_T := K' \cap T(\mathbb{A}_f)$ and $K_T := K \cap T(\mathbb{A}_f)$. Then $K'_T = \prod_p K'_{T,p}$ and $K_T = \prod_p K_{T,p}$. Now we can state our theorem:

Theorem 1. *There exists a constant $B \in (0,1)$ depending only on $(G,\mathcal{X})$ s.t.*

$$|\operatorname{Gal}(\overline{E}/E)s| \geqslant B^{i(T)} |K_T/K'_T| |\operatorname{Gal}(\overline{E}/E)\rho(s)|$$

where $i(T) = |\{p : K_{T,p} \neq K'_{T,p}\}|$.

Proof. This is a direct consequence of Lemma 2, equation (2), Lemma 4 and Lemma 5. □

Remark 1. *This theorem has essentially been studied by Ullmo-Yafaev* [3, §2.2]. *The authors proved this result for a less general $(G,\mathcal{X})$ and a particular K_T, but their proof also works for our $(G,\mathcal{X})$ and arbitrary K_T as long as it is neat. To make the demonstration more clear, we summarize their results and arguments and see how they apply to our $(G,\mathcal{X})$ and a general K_T.*

Lemma 1. *For any point $y \in \mathrm{Sh}_K(G,\mathcal{X})$, K/K' acts (on the right) simply transitively on $\rho^{-1}(y)$.*

Proof. (cf. [3, Lemma 2.11]) Let $y = \overline{(x,g)}$ be a point of $\mathrm{Sh}_K(G,\mathcal{X})$, then $\rho^{-1}(y) = \overline{(x,gK)}$. We first prove

$$\text{For any } a \in K,\ \overline{(x,ga)} = \overline{(x,gak)} \iff k \in K'.$$

The direction $\Leftarrow$ is trivial. Now let us prove $\Rightarrow$. Suppose

$$\overline{(x,ga)} = \overline{(x,gak)} \in \mathrm{Sh}_{K'}(G,\mathcal{X})$$

with $k \in K$. There exist $q \in G(\mathbb{Q})$ and $k' \in K'$ s.t. $x = qx$ and $ga = qgakk'$. By (SV5), we can write $q = q_1 q_2$ (resp. $g = g_1 g_2$) with $q_1 \in Z_G^s(\mathbb{Q}) \simeq (\mathbb{Q}^*)^n$ (resp. $g_1 \in Z_G^s(\mathbb{A}_f) \simeq (\mathbb{A}_f^*)^n$) and $q_2 \in G^c(\mathbb{Q})$ (resp. $g_2 \in G^c(\mathbb{A}_f)$). Now $x = qx$ implies that q_2 is a compact subgroup of $G^c(\mathbb{R})$. The condition $ga = qgakk'$ imples that q_1 (resp. q_2) is in the neat open compact subgroup $g_1(K \cap Z_G^s(\mathbb{A}_f))g_1^{-1}$ (resp. $g_2(K \cap G^c(\mathbb{A}_f))g_2^{-1}$) of $Z_G^s(\mathbb{A}_f) \simeq (\mathbb{A}_f^*)^n$ (resp. $G^c(\mathbb{A}_f)$). But $(\mathbb{Q}^*)^n \cap g_1(K \cap (\mathbb{A}_f^*)^n)g_1^{-1} = 1$, and the intersection of any compact subgroup of $G^c(\mathbb{R})$ with a neat open compact subgroup of $G^c(\mathbb{A}_f)$ is trivial. Hence $q_1 = q_2 = 1$. So $q = 1$. Therefore $k = (k')^{-1} \in K'$.

So K acts transitively on the right on $\rho^{-1}(y)$ and the kernel of this action is K'. So K/K' acts simply transitively on $\rho^{-1}(y)$. □

Lemma 2. $|\operatorname{Gal}(\overline{E}/E)s| \geqslant |\operatorname{Gal}(\overline{E}/E)s \cap \rho^{-1}\rho(s)| \cdot |\operatorname{Gal}(\overline{E}/E)\rho(s)|$.

Proof. (cf. [3, Lemma 2.12]) Because ρ is defined over E, $|\operatorname{Gal}(\overline{E}/E)s \cap \rho^{-1}(\sigma(\rho(s)))|$ is independent of $\sigma \in \operatorname{Gal}(\overline{E}/E)$. This allows us to conclude. □

To give a lower bound for $|\operatorname{Gal}(\overline{E}/E)s \cap \rho^{-1}\rho(s)|$, we shall work with the Shimura subdatum (T, x) of $(G, \mathcal{X})$. The Shimura subdatum (T, x) is defined as follows: $T = \operatorname{MT}(s)$. By [1, Lemma 5.13], $\operatorname{Sh}_{K'}(G, \mathcal{X}) = \coprod \Gamma(g)\backslash\mathcal{X}^+$, where $\Gamma(g) = G(\mathbb{Q})_+ \cap gK'g^{-1}$ is a congruence subgroup of $G(\mathbb{Q})$. Choose $x \in \mathcal{X}^+$ s.t. s is the image of x under the uniformization. It is not hard to check that (T, x) still satisfies (SV5) (see e.g. [3, Remark 2.3]).

Let F be the reflex field of (T, x), then F is a finite extension of E. Define

$$\rho' \colon \operatorname{Sh}_{K'_T}(T, x) \to \operatorname{Sh}_{K_T}(T, x),$$

which is the restriction of ρ, then ρ' is defined over F. It is clear that

$$|\operatorname{Gal}(\overline{E}/E)s \cap \rho^{-1}\rho(s)| \geqslant |\operatorname{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)| \tag{2}$$

Let $\pi_0(\operatorname{Sh}_{K'_T}(T, x))$ be the set of geometric components of $\operatorname{Sh}_{K'_T}(T, x)$. Recall that

$$\pi_0(\operatorname{Sh}_{K'}(T, x)) = T(\mathbb{Q})_+\backslash T(\mathbb{A}_f)/K'_T.$$

This is a finite abelian group. The action of $\operatorname{Gal}(\overline{E}/F)$ on $\pi_0(\operatorname{Sh}_{K'_T}(T, x))$ is given by the reciprocity morphism

$$r \colon \operatorname{Gal}(\overline{E}/F) \to \pi_0(\operatorname{Sh}_{K'_T}(T, x)).$$

Let us describe this action more explicitly. Denote for any $\alpha \in T(\mathbb{A}_f)$ by $\overline{(x, \alpha)}$ the image of (x, α) in $\operatorname{Sh}_{K'_T}(T, x)$. It is a connected component of $\operatorname{Sh}_{K'_T}(T, x)$. As sets we have the following identification:

$$\begin{array}{ccc} \{\overline{(x,\alpha)} \mid \alpha \in T(\mathbb{A}_f)\} & \xrightarrow{\sim} & \pi_0(\operatorname{Sh}_{K'_T}(T, x)) \\ \overline{(x,\alpha)} & \mapsto & \overline{\alpha} \end{array}.$$

Let $\sigma \in \operatorname{Gal}(\overline{E}/F)$ and let $t \in T(\mathbb{A}_f)$ s.t. $\overline{t} = r(\sigma)$, then $\forall \alpha \in T(\mathbb{A}_f)$,

$$\sigma(\overline{(x, \alpha)}) = \overline{(x, t\alpha)} = \overline{(x, \alpha t)}. \tag{3}$$

Recall the following result from Ullmo-Yafaev [3, Proposition 2.9]:

Lemma 3. *There exists a positive integer A depending only on $(G, \mathcal{X})$ s.t. $\forall m \in T(\mathbb{A}_f)$, the image of m^A in $\pi_0(\operatorname{Sh}_{K'_T}(T, x))$ is $r(\sigma)$ for some $\sigma \in \operatorname{Gal}(\overline{E}/F)$.*

Proof. [3, Proposition 2.9], which follows from Lemma 2.4-Lemma 2.8 of *loc.cit.*, announces this result when $Z(G)(\mathbb{R})$ is compact. However the only role this hypothesis plays is to guarantee that $T(\mathbb{Q})$ is discrete (hence closed) in $T(\mathbb{A}_f)$ in Lemma 2.8 of *loc.cit.*. Our hypothesis for $Z(G)$

at the beginning of this section implies that T is an almost product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$ (see e.g. [3, Remark 2.3]), and hence $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$ ([1, Theorem 5.26]). $\square$

Lemma 4. *Let Θ_A be the image of the morphism $k \mapsto k^A$ on K_T/K'_T. We have*

(1) $\Theta_A \cdot s \subset \mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)$;
(2) $|\mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)| \geqslant |\Theta_A|$.

Proof. (cf. [3, Lemma 2.15 & 2.16])

(1) We have $\rho'(\Theta_A \cdot s) = \rho'(s)$. So $\Theta_A \cdot s \subset \rho'^{-1}\rho'(s)$. Moreover similar to Lemma 1, K_H/K'_H acts simply transitively on $\rho'^{-1}\rho'(s)$. For any $\overline{(x,\alpha)} \in \rho'^{-1}\rho'(s)$ and $k \in K_T/K'_T$, this action is given by

$$\overline{(x,\alpha)}k = \overline{(x,\alpha k)}. \tag{4}$$

Let $m \in K_T$, then the image of m^A in $\pi_0(\mathrm{Sh}_{K'_T}(T,x))$ is $r(\sigma)$ for some $\sigma \in \mathrm{Gal}(\overline{E}/F)$ by Lemma 3. It follows that the image of Θ_A in $\pi_0(\mathrm{Sh}_{K'_T}(T,x)) = T(\mathbb{Q})_+\backslash T(\mathbb{A}_f)/K'_T$ is contained in the image of $\mathrm{Gal}(\overline{E}/F)$. So for $s = \overline{(x,\beta)}$, we have by (4) and (3)

$$\Theta_A \cdot s \subset \mathrm{Gal}(\overline{E}/F)s.$$

To sum it up,

$$\Theta_A \cdot s \subset \mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s).$$

(2) By (1) we have

$$|\mathrm{Gal}(\overline{E}/F)s \cap \rho'^{-1}\rho'(s)| \geqslant |\Theta_A \cdot s|.$$

Moreover we have

$$|\rho'^{-1}\rho'(s)| = |(K_T/K'_T) \cdot s| \leqslant \frac{|K_T/K'_T|}{|\Theta_A|}|\Theta_A \cdot s|$$

and

$$|K_T/K'_T| = |\rho'^{-1}\rho'(s)|$$

by the same argument for Lemma 1. These three (in)equalities yield the desired inequality. Remark that we have also proved $|\Theta_A \cdot s| = |\Theta_A|$.

$\square$

Lemma 5. *There exists an integer $r > 0$ depending only on $(G, \mathcal{X})$ s.t.*

$$|\Theta_A| \geqslant \prod_{\{p: K_{T,p} \neq K'_{T,p}\}} \frac{1}{A^r}|K_{T,p}/K'_{T,p}|.$$

Proof. (cf. [3, Lemma 2.18]) Since $K_T/K'_T = \prod_p K_{T,p}/K'_{T,p}$, we have

$$\Theta_A = \prod_{\{p: K_{T,p} \neq K'_{T,p}\}} \Theta_{A,p}.$$

Let L_T be the splitting field of T and let $d := \dim(T)$. $[L_T : \mathbb{Q}]$ is the size of the image of the representation of $\mathrm{Gal}(\overline{E}/\mathbb{Q})$ on the character group $X^*(T)$ of T. This is a finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$ and hence its size is bounded from above in terms of d only. But d is bounded from above in terms of $\dim(G)$ only, so $[L_T : \mathbb{Q}]$ is bounded from above in terms of $\dim(G)$ only.

Using a basis of the character group of T one can embed T into $\mathrm{Res}_{L_T/\mathbb{Q}}\mathbb{G}_{m,L_T}$. Via this embedding, K_T and K'_T are both subgroups of the product of $(\mathbb{Z}_p\otimes O_{L_T})^*$. The group $(\mathbb{Z}_p\otimes O_{L_T})^*$ is the direct product of the groups of units of E_v, completion of E at the place v with $v|p$. By the local unit theorem, the group of units of such an E_v is a dirct product of a cyclic group and $\mathbb{Z}_p^{[E_v:\mathbb{Q}_p]}$.

It follows that there exists a constant r depending only on $(G,\mathcal{X})$ s.t. $K_{T,p}/K'_{T,p}$ is a finite abelian group which is the product of at most r cyclic factors. Therefore the size of the kernel of the A-th power map on $K_{T,p}/K'_{T,p}$ is bounded by A^r, i.e.

$$\Theta_{A,p} \geqslant \frac{1}{A^r}|K_{T,p}/K'_{T,p}|.$$

□

References

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